Positive Energy Unitary Irreducible Representations of the Superalgebras $osp(1|2n,I\!R)$ and Character Formulae[1]

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Abstract

We continue the study of positive energy (lowest weight) unitary irreducible representations of the superalgebras $osp(1|2n,I\!R)$. We update previous results and present the full list of these UIRs. We give also some character formulae for these representations.

1 Introduction

Recently, superconformal field theories in various dimensions are attracting more interest, in particular, due to their duality to AdS supergravities. Until recently only those for $D \leq 6$ were studied since in these cases the relevant superconformal algebras satisfy [1] the Haag-Lopuszanski-Sohnius theorem [2]. Thus, such classification was known only for the $D = 4$ superconformal algebras $su(2,2/N)$ [3] (for $N = 1$), [4–7] (for arbitrary N). More recently, the classification for $D = 3$ (for even N), $D = 5$, and $D = 6$ (for $N =$

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$1, 2)$ was given in [8] (some results are conjectural), and then the $D = 6$ case (for arbitrary N) was finalized in [9].

On the other hand the applications in string theory require the knowledge of the UIRs of the conformal superalgebras for $D > 6$. Most prominent role play the superalgebras $osp(1|\,2n)$. Initially, the superalgebra $osp(1|\,32)$ was put forward for $D = 10$ [10]. Later it was realized that $osp(1|\,2n)$ would fit any dimension, though they are minimal only for $D = 3, 9, 10, 11$ (for $n = 2, 16, 16, 32$, resp.) [11]. In all cases we need to find first the UIRs of $osp(1|\,2n, I\!R)$ which study was started in [12] and [13].

In the present paper we intend to finalize the unitarity classification of [12] and in addition to provide some character formulae.

Since this paper is a sequel of [12], where there is extensive literature, and for the lack of space we only update the supersymmetry literature (for $D > 2$) after 2004, cf. [14–61]

2 Representations of the superalgebras $osp(1|\,2n)$ and $osp(1|\,2n, I\!R)$

2.1 The setting

Our basic references for Lie superalgebras are [62,63], although in this exposition we follow [12].

The even subalgebra of $\mathcal{G} = osp(1|\,2n, I\!R)$ is the algebra $sp(2n, I\!R)$ with maximal compact subalgebra $\mathcal{K} = u(n) \cong su(n) \oplus u(1)$.

We label the relevant representations of $\mathcal{G}$ by the signature:

$$\chi \;=\; [\, d\, ;\, a_1\, , ..., a_{n-1}\,] \tag{1}$$

where d is the conformal weight, and $a_1, ..., a_{n-1}$ are non-negative integers which are Dynkin labels of the finite-dimensional UIRs of the subalgebra $su(n)$ (the simple part of $\mathcal{K}$).

In [12] were classified (with some omissions to be spelled out below) the positive energy (lowest weight) UIRs of $\mathcal{G}$ following the methods used for the $D = 4, 6$ conformal superalgebras, cf. [4–7, 9], resp. The main tool was an adaptation of the Shapovalov form [64] on the Verma modules V^χ over the complexification $\mathcal{G}^{I\!\!C} \;=\; osp(1|\,2n)$ of $\mathcal{G}$.

2.2 Root systems

We recall some facts about $\mathcal{G}^{\mathbb{C}} = osp(1|\,2n)$ (denoted $B(0,n)$ in [62]) as used in [12]. The root systems are given in terms of $\delta_1 \ldots, \delta_n$, $(\delta_i, \delta_j) = \delta_{ij}$, $i, j = 1, ..., n$. The even and odd roots systems are [62]:

$$\begin{aligned} \Delta_{\bar{0}} &= \{\pm\delta_i \pm \delta_j \ , \ 1 \leq i < j \leq n \ , \quad \pm 2\delta_i \ , \ 1 \leq i \leq n\} \ , \\ \Delta_{\bar{1}} &= \{\pm\delta_i \ , \ 1 \leq i \leq n\} \end{aligned} \tag{2}$$

(we remind that the signs $\pm$ are not correlated). We shall use the following distinguished simple root system [62]:

$$\Pi = \{\, \delta_1 - \delta_2 \, , \, , \ldots , \delta_{n-1} - \delta_n \, , \, \delta_n \,\} \ , \tag{3}$$

or introducing standard notation for the simple roots:

$$\begin{aligned} \Pi &= \{\, \alpha_1 \, , ..., \, \alpha_n \,\} \ , \\ &\alpha_j = \delta_j - \delta_{j+1} \ , \quad j = 1, ..., n-1 \ , \quad \alpha_n = \delta_n \ . \end{aligned} \tag{4}$$

The root $\alpha_n = \delta_n$ is odd, the other simple roots are even. The Dynkin diagram is:

$$\underset{1}{\circ} \text{———} \cdots \text{———} \underset{n-1}{\circ} \Longrightarrow \underset{n}{\bullet} \tag{5}$$

The black dot is used to signify that the simple odd root is not nilpotent, otherwise a gray dot would be used [62]. In fact, the superalgebras $B(0,n) = osp(1|\,2n)$ have no nilpotent generators unlike all other types of basic classical Lie superalgebras [62].

The corresponding to Π positive root system is:

$$\Delta^+_{\bar{0}} = \{\delta_i \pm \delta_j \, , \, 1 \leq i < j \leq n \, , \ 2\delta_i \, , \, 1 \leq i \leq n\} \, , \qquad \Delta^+_{\bar{1}} = \{\delta_i \, , \, 1 \leq i \leq n\} \tag{6}$$

We record how the elementary functionals are expressed through the simple roots:

$$\delta_k = \alpha_k + \cdots + \alpha_n \ . \tag{7}$$

From the point of view of representation theory more relevant is the restricted root system, such that:

$$\bar{\Delta}^+ = \bar{\Delta}^+_{\bar{0}} \cup \Delta^+_{\bar{1}} \ , \tag{8}$$

$$\bar{\Delta}^+_{\bar{0}} \equiv \{\alpha \in \Delta^+_{\bar{0}} \mid \tfrac{1}{2}\alpha \notin \Delta^+_{\bar{1}}\} = \{\delta_i \pm \delta_j \ , \ 1 \leq i < j \leq n\} \tag{9}$$

The superalgebra $\mathcal{G} = osp(1|2n, I\!R)$ is a split real form of $osp(1|2n)$ and has the same root system.

The above simple root system is also the simple root system of the complex simple Lie algebra B_n (dropping the distinction between even and odd roots) with Dynkin diagram:

$$\underset{1}{\circ} \text{——} \cdots \text{——} \underset{n-1}{\circ} \Longrightarrow \underset{n}{\circ} \tag{10}$$

Naturally, for the B_n positive root system we drop the roots $2\delta_i$

$$\Delta^+_{\mathrm{B_n}} = \{\delta_i \pm \delta_j \ , \ 1 \leq i < j \leq n \ , \ \ \delta_i \ , \ 1 \leq i \leq n\} \cong \bar{\Delta}^+ \tag{11}$$

This shall be used essentially below.

2.3 Lowest weight through the signature

Besides (1) we shall use the Dynkin-related labelling:

$$(\Lambda, \alpha_k^\vee) = -\, a_k \ , \quad 1 \leq k \leq n \ , \tag{12}$$

where $\alpha_k^\vee \equiv 2\alpha_k/(\alpha_k, \alpha_k)$, and the minus signs are related to the fact that we work with lowest weight Verma modules (instead of the highest weight modules used in [63]) and to Verma module reducibility w.r.t. the roots α_k (this is explained in detail in [6, 12]).

Obviously, a_n must be related to the conformal weight d which is a matter of normalization so as to correspond to some known cases. Thus, our choice is:

$$a_n = -\, 2d - a_1 - \cdots - a_{n-1} \ . \tag{13}$$

The actual Dynkin labelling is given by:

$$m_k = (\rho - \Lambda, \alpha_k^\vee) \tag{14}$$

where $\rho \in \mathcal{H}^*$ is given by the difference of the half-sums $\rho_{\bar{0}}\,, \rho_{\bar{1}}$ of the even, odd, resp., positive roots (cf. (6):

$$\begin{aligned} \rho \quad &\doteq \quad \rho_{\bar{0}} - \rho_{\bar{1}} = (n - \tfrac{1}{2})\delta_1 + (n - \tfrac{3}{2})\delta_2 + \cdots + \tfrac{3}{2}\delta_{n-1} + \tfrac{1}{2}\delta_n \ , \\ \rho_{\bar{0}} &= n\delta_1 + (n-1)\delta_2 + \cdots + 2\delta_{n-1} + \delta_n \ , \\ \rho_{\bar{1}} &= \tfrac{1}{2}(\delta_1 + \cdots + \delta_n) \ . \end{aligned} \tag{15}$$

Naturally, the value of ρ on the simple roots is 1: $(\rho, \alpha_i^\vee) = 1$, $i = 1, ..., n$.

Unlike $a_k \in \mathbb{Z}_+$ for $k < n$ the value of a_n is arbitrary. In the cases when a_n is also a non-negative integer, and then $m_k \in \mathbb{N}$ $(\forall k)$ the corresponding irreps are the finite-dimensional irreps of $\mathcal{G}$ (and of B_n).

Having in hand the values of Λ on the basis we can recover them for any element of $\mathcal{H}^*$.

We shall need only $(\Lambda, \beta^\vee)$ for all positive roots β as given in [12]:

$$\begin{aligned}
(\Lambda, (\delta_i - \delta_j)^\vee) &= (\Lambda, \delta_i - \delta_j) = -a_i - \cdots - a_{j-1} \\
(\Lambda, (\delta_i + \delta_j)^\vee) &= (\Lambda, \delta_i + \delta_j) = 2d + a_1 + \cdots + a_{i-1} - a_j - \cdots - a_{n-1} \\
(\Lambda, \delta_i^\vee) &= (\Lambda, 2\delta_i) = 2d + a_1 + \cdots + a_{i-1} - a_i - \cdots - a_{n-1} \\
(\Lambda, (2\delta_i)^\vee) &= (\Lambda, \delta_i) = d + \tfrac{1}{2}(a_1 + \cdots + a_{i-1} - a_i - \cdots - a_{n-1})
\end{aligned} \tag{16}$$

2.4 Verma modules

To introduce Verma modules we use the standard triangular decomposition:

$$\mathcal{G}^{\mathbb{C}} = \mathcal{G}^+ \oplus \mathcal{H} \oplus \mathcal{G}^- \tag{17}$$

where $\mathcal{G}^+$, $\mathcal{G}^-$, resp., are the subalgebras corresponding to the positive, negative, roots, resp., and $\mathcal{H}$ denotes the Cartan subalgebra.

We consider lowest weight Verma modules, so that $V^\Lambda \cong U(\mathcal{G}^+) \otimes v_0$, where $U(\mathcal{G}^+)$ is the universal enveloping algebra of $\mathcal{G}^+$, and v_0 is a lowest weight vector v_0 such that:

$$\begin{aligned}
Z\ v_0 &= 0\ , \quad Z \in \mathcal{G}^- \\
H\ v_0 &= \Lambda(H)\ v_0\ , \quad H \in \mathcal{H}\ .
\end{aligned} \tag{18}$$

Further, for simplicity we omit the sign $\otimes$, i.e., we write $p\, v_0 \in V^\Lambda$ with $p \in U(\mathcal{G}^+)$.

Adapting the criterion of [63] (which generalizes the BGG-criterion [65] to the super case) to lowest weight modules, one finds that a Verma module V^Λ is reducible w.r.t. the positive root β iff the following holds [12]:

$$(\rho - \Lambda, \beta^\vee) = m_\beta\ , \qquad \beta \in \Delta^+\ , \quad m_\beta \in \mathbb{N}\ . \tag{19}$$

If a condition from (19) is fulfilled then V^Λ contains a submodule which is a Verma module $V^{\Lambda'}$ with shifted weight given by the pair m, β : $\Lambda' = \Lambda + m\beta$. The embedding of $V^{\Lambda'}$ in V^Λ is provided by mapping the lowest weight

vector v'_0 of $V^{\Lambda'}$ to the **singular vector** $v_s^{m,\beta}$ in V^Λ which is completely determined by the conditions:

$$\begin{aligned} X\ v_s^{m,\beta} &= 0\,, \quad X \in \mathcal{G}^-\,, \\ H\ v_s^{m,\beta} &= \Lambda'(H)\ v_0\,, \quad H \in \mathcal{H}\,, \quad \Lambda' = \Lambda + m\beta\,. \end{aligned} \tag{20}$$

Explicitly, $v_s^{m,\beta}$ is given by a polynomial in the positive root generators [6,66]:

$$v_s^{m,\beta} = P^{m,\beta}\, v_0\,, \quad P^{m,\beta} \in U(\mathcal{G}^+)\,. \tag{21}$$

Thus, the submodule I^β of V^Λ which is isomorphic to $V^{\Lambda'}$ is given by $U(\mathcal{G}^+)\,P^{m,\beta}\,v_0$.

Note that the Casimirs of $\mathcal{G}^{\mathbb{C}}$ take the same values on V^Λ and $V^{\Lambda'}$.

Certainly, (19) may be fulfilled for several positive roots (even for all of them). Let Δ_Λ denote the set of all positive roots for which (19) is fulfilled, and let us denote: $\tilde{I}^\Lambda \equiv \cup_{\beta\in\Delta_\Lambda} I^\beta$. Clearly, $\tilde{I}^\Lambda$ is a proper submodule of V^Λ. Let us also denote $F^\Lambda \equiv V^\Lambda/\tilde{I}^\Lambda$.

Further we shall use also the following notion. The singular vector v_1 is called **descendant** of the singular vector $v_2 \notin \mathbb{C}v_1$ if there exists a homogeneous polynomial P_{12} in $U(\mathcal{G}^+)$ so that $v_1 = P_{12}\,v_2$. Clearly, in this case we have: $I^1 \subset I^2$, where I^k is the submodule generated by v_k.

The Verma module V^Λ contains a unique proper maximal submodule I^Λ ($\supseteq \tilde{I}^\Lambda$) [63,65]. Among the lowest weight modules with lowest weight Λ there is a unique irreducible one, denoted by L_Λ, i.e., $L_\Lambda = V^\Lambda/I^\Lambda$. (If V^Λ is irreducible then $L_\Lambda = V^\Lambda$.)

It may happen that the maximal submodule I^Λ coincides with the submodule $\tilde{I}^\Lambda$ generated by all singular vectors. This is, e.g., the case for all Verma modules if rank $\mathcal{G} \leq 2$, or when (19) is fulfilled for all simple roots (and, as a consequence for all positive roots). Here we are interested in the cases when $\tilde{I}^\Lambda$ is a proper submodule of I^Λ. We need the following notion.

Definition: [65,67,68] *Let V^Λ be a reducible Verma module. A vector $v_{\rm ssv} \in V^\Lambda$ is called a* **subsingular vector** *if $v_{\rm su} \notin \tilde{I}^\Lambda$ and the following holds:*

$$X\ v_{\rm su} \in \tilde{I}^\Lambda\,, \quad \forall X \in \mathcal{G}^- \tag{22}$$

Going from the above more general definitions to $\mathcal{G}$ we recall that in [12] it was established that from (19) follows that the Verma module $V^{\Lambda(\chi)}$ is

reducible if one of the following relations holds (following the order of (16):

$$\begin{aligned}
I\!\!N \ni m^-_{ij} &= j - i + a_i + \cdots + a_{j-1} && \text{(23a)}\\
I\!\!N \ni m^+_{ij} &= 2n - i - j + 1 + a_j + \cdots + a_{n-1} - a_1 - \cdots - a_{i-1} - 2d && \text{(23b)}\\
I\!\!N \ni m_i &= 2n - 2i + 1 + a_i + \cdots + a_{n-1} - a_1 + \cdots - a_{i-1} - 2d && \text{(23c)}\\
I\!\!N \ni m_{ii} &= n - i + \tfrac{1}{2}(1 + a_i + \cdots + a_{n-1} - a_1 + \cdots - a_{i-1}) - d\ . && \text{(23d)}
\end{aligned}$$

Further we shall use the fact from [12] that we may eliminate the reducibilities and embeddings related to the roots $2\delta_i$. Indeed, since $m_i = 2m_{ii}$, whenever (23d) is fulfilled also (23c) is fulfilled.

For further use we introduce notation for the root vector $X^+_j \in \mathcal{G}^+$, $j = 1, \ldots, n,$ corresponding to the simple root α_j. Naturally, $X^-_j \in \mathcal{G}^-$ corresponds to $-\alpha_j$.

Further, we notice that all reducibility conditions in (23a) are fulfilled. In particular, for the simple roots from those condition (23a) is fulfilled with $\beta \to \alpha_i = \delta_i - \delta_{i+1}$, $i = 1, ..., n-1$ and $m^-_i \equiv m^-_{i,i+1} = 1 + a_i$. The corresponding submodules $I^{\alpha_i} = U(\mathcal{G}^+)\, v^i_s$, where $\Lambda_i = \Lambda + m^-_i \alpha_i$ and $v^i_s = (X^+_i)^{1+a_i}\, v_0$. These submodules generate an invariant submodule which we denote by $I^\Lambda_c \subset \tilde{I}^\Lambda$. Since these submodules are nontrivial for all our signatures in the question of unitarity instead of V^Λ we shall consider also the factor-modules:

$$F^\Lambda_c = V^\Lambda / I^\Lambda_c \supset F^\Lambda\ . \tag{24}$$

We shall denote the lowest weight vector of F^Λ_c by $|\Lambda_c\rangle$ and the singular vectors above become null conditions in F^Λ_c :

$$(X^+_i)^{1+a_i}\, |\Lambda_c\rangle = 0\ , \quad i = 1, ..., n-1. \tag{25}$$

If the Verma module V^Λ is not reducible w.r.t. the other roots, i.e., (23b,c,d) are not fulfilled, then $F^\Lambda_c = F^\Lambda$ is irreducible and is isomorphic to the irrep L_Λ with this weight.

In fact, for the factor-modules reducibility is controlled by the value of d, or in more detail:

The maximal d coming from the different possibilities in (23b) are obtained for $m^+_{ij} = 1$ and they are:

$$d_{ij} \equiv n + \tfrac{1}{2}(a_j + \cdots + a_{n-1} - a_1 - \cdots - a_{i-1} - i - j)\ , \tag{26}$$

the corresponding root being $\delta_i + \delta_j$.

The maximal d coming from the different possibilities in (23c,d), resp., are obtained for $m_i = 1$, $m_{ii} = 1$, resp., and they are:

$$\begin{aligned} d_i &\equiv n - i + \tfrac{1}{2}(a_i + \cdots + a_{n-1} - a_1 - \cdots - a_{i-1}) \ , \\ d_{ii} &= d_i - \tfrac{1}{2} \ , \end{aligned} \tag{27}$$

the corresponding roots being $\delta_i \, , 2\delta_j$, resp.

There are some orderings between these maximal reduction points [12]:

$$\begin{aligned} d_1 &> d_2 > \cdots > d_n \ , \\ d_{i,i+1} &> d_{i,i+2} > \cdots > d_{in} \ , \\ d_{1,j} &> d_{2,j} > \cdots > d_{j-1,j} \ , \\ d_i &> d_{jk} > d_\ell \ , \qquad i \leq j < k \leq \ell \ . \end{aligned} \tag{28}$$

Obviously the first reduction point is:

$$d_1 = n - 1 + \tfrac{1}{2}(a_1 + \cdots + a_{n-1}) \ . \tag{29}$$

3 Unitarity

The first results on the unitarity were given in [12]. These were not complete so the statement below should be called *Dobrev-Zhang-Salom Theorem.*

Theorem: All positive energy unitary irreducible representations of the superalgebras $osp(1|\,2n, I\!\!R)$ characterized by the signature χ in (1) are

obtained for real d and are given as follows:

$$\begin{aligned}
d &\geq n-1+\tfrac{1}{2}(a_1+\cdots+a_{n-1}) = d_1 , \quad a_1\neq 0 , && (30)\\
d &\geq n-\tfrac{3}{2}+\tfrac{1}{2}(a_2+\cdots+a_{n-1}) = d_{12} , \quad a_1=0,\ a_2\neq 0 , &&\\
d &= n-2+\tfrac{1}{2}(a_2+\cdots+a_{n-1}) = d_2 > d_{13} , \quad a_1=0,\ a_2\neq 0 , && (31)\\
d &\geq n-2+\tfrac{1}{2}(a_3+\cdots+a_{n-1}) = d_2 = d_{13} , \quad a_1=a_2=0,\ a_3\neq 0 , &&\\
d &= n-\tfrac{5}{2}+\tfrac{1}{2}(a_3+\cdots+a_{n-1}) = d_{23} > d_{14} , \quad a_1=a_2=0,\ a_3\neq 0 , &&\\
d &= n-3+\tfrac{1}{2}(a_3+\cdots+a_{n-1}) = d_3 = d_{24} > d_{15} , \quad a_1=a_2=0,\ a_3\neq 0 , &&\\
&\ldots &&\\
&\ldots &&\\
d &\geq n-1-\kappa+\tfrac{1}{2}(a_{2\kappa+1}+\cdots+a_{n-1}) , \quad a_1=\ldots=a_{2\kappa}=0,\ a_{2\kappa+1}\neq 0 , &&\\
&\qquad \kappa=\tfrac{1}{2},1,\ldots,\tfrac{1}{2}(n-1) , &&\\
d &= n-\tfrac{3}{2}-\kappa+\tfrac{1}{2}(a_{2\kappa+1}+\cdots+a_{n-1}) , \quad a_1=\ldots=a_{2\kappa}=0,\ a_{2\kappa+1}\neq 0 , &&\\
&\ldots &&\\
d &= n-1-2\kappa+\tfrac{1}{2}(a_{2\kappa+1}+\cdots+a_{n-1}) , \quad a_1=\ldots=a_{2\kappa}=0,\ a_{2\kappa+1}\neq 0 , &&\\
&\ldots &&\\
&\ldots &&\\
d &\geq \tfrac{1}{2}(n-1) , \quad a_1 = \ldots = a_{n-1} = 0 &&\\
d &= \tfrac{1}{2}(n-2) , \quad a_1 = \ldots = a_{n-1} = 0 &&\\
&\ldots &&\\
d &= \tfrac{1}{2} , \quad a_1 = \ldots = a_{n-1} = 0 &&\\
d &= 0 , \quad a_1 = \ldots = a_{n-1} = 0 &&
\end{aligned}$$

Proof: The statement of the Theorem for $d > d_1$ is clear in [12] from the general considerations since this is the First reduction point. For $d = d_1$ (also following [12]) we have the first zero norm state which is naturally given by the corresponding singular vector $v^1_{\delta_1} = \mathcal{P}^{1,\delta_1} v_0$. In fact, all states of the embedded submodule $V^{\Lambda+\delta_1}$ built on $v^1_{\delta_1}$ have zero norms. Due to the above singular vector we have the following additional null condition in F_c^Λ :

$$\mathcal{P}^{1,\delta_1} \, \widetilde{|\Lambda\rangle} = 0 . \tag{32}$$

The above condition factorizes the submodule built on $v^1_{\delta_1}$. There are no other vectors with zero norm at $d = d_1$ since by a general result [63], the elementary embeddings between Verma modules are one-dimensional. Thus,

F^Λ is the UIR $L_\Lambda = F^\Lambda$.
Below $d < d_1$ there is no unitarity for $a_1 \neq 0$. On the other hand (as shown in [12]) for $a_1 = 0$ the singular vector $v^1_{\delta_1}$ is descendant of the compact root singular vector $X_1^+ v_0$ which is already factored out for $a_1 = 0$. Thus, below we set $a_1 = 0$.

The next reducibility point is $d = d_{12} = n - \frac{3}{2} + \frac{1}{2}(a_2 + \cdots + a_{n-1})$. The corresponding root is $\delta_1 + \delta_2 = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n$. The corresponding singular vector is $v^1_{\delta_1+\delta_2} = \mathcal{P}^{1,\delta_1+\delta_2} v_0$. All states of the embedded submodule $V^{\Lambda+\delta_1+\delta_2}$ built on $v^1_{\delta_1+\delta_2}$ have zero norms for $d = d_{12}$. Due to the above singular vector we have the following additional null condition in F^Λ_c :

$$\mathcal{P}^{1,\delta_1+\delta_2} \widetilde{|\Lambda\rangle} = 0 , \qquad d = d_{12} . \tag{33}$$

The above conditions factorizes the submodule built on $v^1_{\delta_1+\delta_2}$. Thus, F^Λ_c is the UIR $L_\Lambda = F^\Lambda_c$.

Below $d < d_{12}$ there is no unitarity for $a_2 \neq 0$, except at the isolated point: $d_2 = n - 2 + \frac{1}{2}(a_2 + \cdots + a_{n-1})$. At the latter point there is a singular vector $v^1_{\delta_2}$ which must be factored for unitarity. In addition, the previous singular vector is descendant of $v^1_{\delta_2}$ and the compact root singular vector $X_1^+ v_0$.

Further, for for $a_2 = 0$ the singular vectors $v^1_{\delta_1+\delta_2}$ and $v^1_{\delta_2}$ are descendants of the compact root singular vectors $X_1^+ v_0$ and $X_2^+ v_0$ which are factored out for $a_1 = a_2 = 0$. Thus, below we set also $a_2 = 0$ and there would be no obstacles for unitarity until the next reducibility points (coinciding due $a_2 = 0$): $d_2 = d_{13} = n - 2 + \frac{1}{2}(a_3 + \cdots + a_{n-1})$. The singular vector for $d = d_{13}$ and $m = 1$ has weight $\delta_1 + \delta_3 = \alpha_1 + \alpha_2 + 2\alpha_3 + \cdots 2\alpha_n$ and for $a_1 = 0$ it is a descendant of the compact root singular vector $X_1 v_0$ [70]. However, at $d_2 = d_{13}$ there is a subsingular vector which must be factored for unitarity. For $d < d_2 = d_{13}$ and $a_3 \neq 0$ the norm of that subsingular vector is negative, and there will be no unitarity except at some lower reducibility points.

For $d_{23} = n - \frac{5}{2} + \frac{1}{2}(a_3 + \cdots + a_{n-1})$ there is singular vector $v^1_{\delta_2+\delta_3}$ of weight $\delta_2 + \delta_3 = \alpha_2 + 2\alpha_3 + \cdots 2\alpha_n$ [70] which must be factored for unitarity. The previous subsingular vector is also factored out since it is descendant of $v^1_{\delta_2+\delta_3}$ and compact root singular vectors.

Further on, the Proof goes on similar lines. We list the points at which there are subsingular vectors - these happen when reducibility points coincide

due the zero values of some a_i :

$$
\begin{aligned}
& d_2 = d_{13} = n-2+\tfrac{1}{2}(a_3+\cdots+a_{n-1}) \ , \quad a_1=a_2=0 \ , \qquad (34)\\
& d_{23} = d_{14} = n-5/2+\tfrac{1}{2}(a_4+\cdots+a_{n-1}) \ , \quad a_1=a_2=a_3=0, \ n>3,\\
& d_3 = d_{24} = d_{15} = n-3+\tfrac{1}{2}(a_5+\cdots+a_{n-1}) \ , \quad a_1=a_2=a_3=a_4=0,\\
& \qquad\qquad n>3,\\
& \ldots\\
& d_j = d_{1,2j-1} = d_{2,2j-2} = \cdots = d_{j-1,j+1} = n-j+\tfrac{1}{2}(a_{2j-1}+\cdots+a_{n-1}) \ ,\\
& \qquad a_1=\cdots=a_{2j-2}=0, \quad j<n,\\
& d_{j,j+1} = d_{1,2j} = d_{2,2j-1} = \cdots = d_{j-1,j+2} = n-j-\tfrac{1}{2}+\tfrac{1}{2}(a_{2j}+\cdots+a_{n-1}) \ ,\\
& \qquad a_1=\cdots=a_{2j-1}=0, \quad j<n-1.
\end{aligned}
$$

Above it is understood that $a_j \equiv 0$ for $j \geq n$.
At the points of the subsingular vectors the associated singular vectors are factored out automatically. This happens also when the subsingular vectors are inside a continuous part of the unitarity spectrum. ■

The Proof above is not as explicit as we would like it to be, but due to the lack of space we postpone it to [74]. Below we give separately and explicitly the case $n=3$.

Example: n=3. For $n=3$ f-la (28) simplifies to:

(35)

The Theorem now reads:

$$
\begin{aligned}
d &\geq 2+\tfrac{1}{2}(a_1+a_2) = d_1 \ , \quad a_1 \neq 0 \ , \qquad (36)\\
d &\geq \tfrac{3}{2}+\tfrac{1}{2}a_2 = d_{12} \ , \quad a_1=0, \ a_2\neq 0 \ ,\\
d &= 1+\tfrac{1}{2}a_2 = d_2 > d_{13} \ , \quad a_1=0, \ a_2\neq 0 \ ,\\
d &\geq 1 = d_2 = d_{13} \ , \quad a_1=a_2=0 \ ,\\
d &= \tfrac{1}{2} = d_{23} \ , \quad a_1=a_2=0 \ ,\\
d &= 0 = d_3 \ , \quad a_1=a_2=0 \ .
\end{aligned}
$$

For $d > d_1$ there are no singular vectors and we have unitarity. At $d = d_1$ there is a singular vector:

$$v^1_{\delta_1} = \Big(a_1(a_1+a_2+1)X_{\delta_1} - a_1 X_{\delta_3}\, X_{13} - (a_1+a_2+1)X_{\delta_2}\, X_1^+ + \\ + X_{\delta_3}\, X_2^+\, X_1^+\Big)\, v_0 \qquad (37)$$

which is given in PBW basis, where $X_{\delta_j} \in \mathcal{G}^+$ are the vectors corresponding to the weight vectors δ_j, X_{13} is the compact root vector for $\alpha_{13} = \alpha_1 + \alpha_2 = \delta_1 - \delta_3$. This singular vector is non-trivial for $a_1 \neq 0$ and must be eliminated to obtain an UIR. Below $d < d_1$ there is no unitarity for $a_1 \neq 0$. On the other hand for $a_1 = 0$ the singular vector $v^1_{\delta_1}$ is descendant of the compact root singular vector $X_1^+ v_0$ which is already factored out for $a_1 = 0$. Thus, below we discuss only the cases with $a_1 = 0$.
The singular vector at $d = d_{12}$ corresponding to the root $\delta_1 + \delta_2 = \alpha_1 + 2\alpha_2 + 2\alpha_3$ is:

$$\begin{aligned} v^1_{\delta_1+\delta_2} = \Big(& X_{\delta_3} X_{\delta_2} X_2^+ X_1^+ + \frac{1}{2}(X_{\delta_3})^2 (X_2^+)^2 X_1^+ - a_1 (X_{\delta_3})^2 X_2^+ X_{13} + \\ & + 2(a_2+1)\; X_{\delta_3} X_{\delta_2} X_{13} - 2(a_1+a_2+1)\; X_{\delta_3} X_{\delta_1} X_2^+ + \\ & + (a_1+1)(a_1+a_2+1)\; X_{\delta_1+\delta_3} X_2^+ + 4a_2 (a_1+a_2+1)\; X_{\delta_2} X_{\delta_1} + \\ & + 2a_2 (a_1+a_2+1)\; (X_{\delta_2})^2 X_1^+ - \frac{1}{2}(a_1+2a_2+1)\; X_{\delta_2+\delta_3} X_2^+ X_1^+ - \\ & - (-a_2 a_1 + a_1 + a_2 + 1)\; X_{\delta_2+\delta_3} X_{13} - \\ & - 2(a_1+1)\, a_2\, (a_1+a_2+1)\, X_{\delta_1+\delta_2} \Big) v_0 \qquad (38) \end{aligned}$$

with norm:

$$16\,(2d - a_2 - 3)\left(a_1^2 + 2a_1 + 2d - a_2 - 2\right) a_2\,(a_2+1)\,(a_1+a_2+1)\,(a_1+a_2+2)\,.$$

For $a_1 = 0$, $a_2 \neq 0$ it is non-trivial and gives rise to a invariant subspace which must be factored out for unitarity. For $d < d_{12}$ the vector (38) has negative norm and there is no unitarity for $a_2 \neq 0$, except at the isolated unitary point $d = 1 + \frac{1}{2}a_2 = d_2 > d_{13}$. At this point there is a singular vector v_{s2}, while the vector (38) is descendant of compact root singular vector $X_1 v_0$ and v_{s2}.
Further, we consider $a_1 = a_2 = 0$. Then the vector $v^1_{\delta_1+\delta_2}$ is descendant of compact root singular vectors $X_1^+ v_0$ and $X_2^+ v_0$, thus, there is no obstacle

for unitarity for $1 < d$. The next reducibility points (coinciding here) are $d = d_{13} = d_2 = 1$. The singular vector for $d = d_2$ and $m = 1$ has weight $\delta_2 = \alpha_2 + \alpha_3$ and is given by:

$$v^1_{\delta_2} \ = \ \Big(a_2 X_2^+ X_3^+ - (a_2+1) X_3^+ X_2^+\Big) v_0 \tag{39}$$

For $a_2 = 0$ it is a descendant of the compact root singular vector $X_2^+\, v_0$. The singular vector for $d = d_{13} = 1$ and $m = 1$ has weight $\delta_1 + \delta_3 = a_1 + \alpha_2 + 2\alpha_3$ [70] :

$$\begin{aligned} v^1_{\delta_1+\delta_3} \ &= \ \Big(h a_1\, X_1^+ (X_3^+)^2 X_2^+ + a_1\, X_1^+ X_3^+ X_2^+ X_3^+ - \\ &- h(a_1+1)(X_3^+)^2 X_2^+ X_1^+ \ - \\ &- h a_1\, X_1^+ X_2^+ (X_3^+)^2 - (a_1+1) X_3^+ X_2^+ X_3^+ X_1^+ \ + \\ &+ h(a_1+1) X_2^+ (X_3^+)^2 X_1^+ \Big), \\ &\qquad h = 1 + \tfrac{1}{2}(a_1 + a_2) \end{aligned} \tag{40}$$

The above vector is given in the simple root basis most appropriate for the case. For $a_1 = 0$ it is a descendant of the compact root singular vector $X_1^+\, v_0$. However, there is a subsingular vector:

$$v_{ss} \ = \ (X_{\delta_1} X_{\delta_2} X_{\delta_3} - X_{\delta_3} X_{\delta_2} X_{\delta_1}) v_0 \tag{41}$$

with norm: $16d(d-1)(2d-1)$. This must be factorized in order to obtain UR. Then for $\frac{1}{2} < d < 1$ there will be no unitarity due to the last norm.

Finally, at the next reducibility point: $d = d_{23} = \frac{1}{2}$ there is a singular vector of weight $\delta_2 + \delta_3 = \alpha_2 + 2\alpha_3$:

$$v^1_{\delta_2+\delta_3} \ = \ (2X_{\delta_2+\delta_3} - 4X_{\delta_2} X_{\delta_3} + X_{2\delta_3} X_2^+) v_0 \tag{42}$$

It should be factored out to get unitarity. The subsingular vector (41) has zero norm for $d = \frac{1}{2}$ and furthermore it is descendant of $v^1_{\delta_2+\delta_3}$ and the compact root singular vector $X_2^+\, v_0$. Finally, for $d < \frac{1}{2}$ there is no unitarity since then the norm of (42) is negative, except at the trivial isolated unitary point $d = 0 = a_1 = a_2$ of one-dimensional irrep. ■

4 Character formulae

4.1 Character formulae: generalities

In the beginning of this subsection we follow [73]. Let $\hat{\mathcal{G}}$ be a simple Lie algebra of rank ℓ with Cartan subalgebra $\hat{\mathcal{H}}$, root system $\hat{\Delta}$, simple root system $\hat{\pi}$. Let Γ, (resp. Γ_+), be the set of all integral, (resp. integral dominant), elements of $\hat{\mathcal{H}}^*$, i.e., $\lambda \in \hat{\mathcal{H}}^*$ such that $(\lambda, \alpha_i^\vee) \in \mathbb{Z}$, (resp. $\mathbb{Z}_+$), for all simple roots α_i , $(\alpha_i^\vee \equiv 2\alpha_i/(\alpha_i, \alpha_i))$. Let V be a lowest weight module with lowest weight Λ and lowest weight vector v_0 . It has the following decomposition:

$$V \ = \ \underset{\mu\in\Gamma_+}{\oplus} V_\mu \ \ , \qquad V_\mu \ = \ \{u \in V \mid Hu = (\lambda+\mu)(H)u, \ \forall\ H \in \mathcal{H}\} \quad (43)$$

(Note that $V_0 = \mathbb{C} v_0$.) Let $E(\mathcal{H}^*)$ be the associative abelian algebra consisting of the series $\sum_{\mu\in\mathcal{H}^*} c_\mu e(\mu)$, where $c_\mu \in \mathbb{C}$, $c_\mu = 0$ for μ outside the union of a finite number of sets of the form $D(\lambda) = \{\mu \in \mathcal{H}^* | \mu \geq \lambda\}$, using some ordering of $\mathcal{H}^*$, e.g., the lexicographic one; the formal exponents $e(\mu)$ have the properties: $e(0) = 1$, $e(\mu)e(\nu) = e(\mu + \nu)$.

Then the (formal) character of V is defined by:

$$ch_0\ V \ \ = \ \ \sum_{\mu\in\Gamma_+} (\dim\ V_\mu)\ e(\Lambda+\mu) \ \ = \ \ e(\Lambda) \sum_{\mu\in\Gamma_+} (\dim\ V_\mu)\ e(\mu) \qquad (44)$$

(We shall use subscript '0' for the even case.)

For a Verma module, i.e., $V = V^\Lambda$ one has dim $V_\mu = P(\mu)$, where $P(\mu)$ is a generalized partition function, $P(\mu) = \#$ of ways μ can be presented as a sum of positive roots β, each root taken with its multiplicity $\dim \mathcal{G}_\beta$ $(= 1$ here), $P(0) \equiv 1$. Thus, the character formula for Verma modules is:

$$ch_0\ V^\Lambda \ \ = \ \ e(\Lambda) \sum_{\mu\in\Gamma_+} P(\mu) e(\mu) \ \ = \ \ e(\Lambda) \prod_{\alpha\in\Delta^+} (1 - e(\alpha))^{-1}. \qquad (45)$$

Further we recall the standard reflections in $\hat{\mathcal{H}}^*$:

$$s_\alpha(\lambda) \ = \ \lambda - (\lambda, \alpha^\vee)\alpha\,, \quad \lambda \in \hat{\mathcal{H}}^*\,, \quad \alpha \in \hat{\Delta}. \qquad (46)$$

The Weyl group W is generated by the simple reflections $s_i \equiv s_{\alpha_i}$, $\alpha_i \in \hat{\pi}$. Thus every element $w \in W$ can be written as the product of simple

reflections. It is said that w is written in a reduced form if it is written with the minimal possible number of simple reflections; the number of reflections of a reduced form of w is called the length of w, denoted by $\ell(w)$.

The Weyl character formula for the finite-dimensional irreducible LWM L_Λ over $\hat{\mathcal{G}}$, i.e., when $\Lambda \in -\Gamma_+$, has the form:

$$ch_0\ L_\Lambda \ = \ \sum_{w\in W} (-1)^{\ell(w)}\ ch_0\ V^{w\cdot\Lambda}\,, \quad \Lambda \in -\Gamma_+ \tag{47}$$

where the dot $\cdot$ action is defined by $w\cdot\lambda = w(\lambda-\rho)+\rho$. For future reference we note:

$$s_\alpha \ \cdot\ \Lambda \ = \ \Lambda \ + \ n_\alpha \alpha \tag{48}$$

where

$$n_\alpha \ = \ n_\alpha(\Lambda) \ \doteq \ (\rho - \Lambda, \alpha^\vee) \ = \ (\rho-\Lambda)(H_\alpha)\,, \quad \alpha \in \Delta^+. \tag{49}$$

In the case of basic classical Lie superalgebras the first character formulae were given by Kac [63, 71].[2] For all such superalgebras – except $osp(1/2n)$ – the character formula for Verma modules is [63, 71]:

$$ch\ V^\Lambda \ = \ e(\Lambda) \left(\prod_{\alpha\in\Delta^+_{\bar 0}} (1-e(\alpha))^{-1} \right) \left(\prod_{\alpha\in\Delta^+_{\bar 1}} (1+e(\alpha)) \right). \tag{50}$$

We are however interested exactly in the $osp(1/2n)$ when the Verma module character formula is:

$$ch\ V^\Lambda \ = \ e(\Lambda) \left(\prod_{\alpha\in\bar\Delta^+} (1-e(\alpha))^{-1} \right) \tag{51}$$

Naturally, the character formula for the finite-dimensional irreducible LWM L_Λ is again (47) using the Weyl group W_n of B_n.

4.2 Multiplets

A Verma module V^Λ may be reducible w.r.t. to many positive roots, and thus there maybe many Verma modules isomorphic to its submodules. They themselves may be reducible, and so on.

[2] Kac considers highest weight modules but his results are immediately transferable to lowest weight modules.

One main ingredient of the approach of [66] is as follows. We group the (reducible) Verma modules with the same Casimirs in sets called *multiplets* [69]. The multiplet corresponding to fixed values of the Casimirs may be depicted as a connected graph, the vertices of which correspond to the reducible Verma modules and the lines between the vertices correspond to embeddings between them. The explicit parametrization of the multiplets and of their Verma modules is important for understanding of the situation.

If a Verma module V^Λ is reducible w.r.t. to all simple roots (and thus w.r.t. all positive roots), i.e., $m_k \in I\!N$ for all k, then the irreducible submodules are isomorphic to the finite-dimensional irreps of $\mathcal{G}^{I\!\!C}$ [66]. (Actually, this is a condition only for m_n since $m_k \in I\!N$ for $k = 1, \ldots, n-1$.) In these cases we have the *main multiplets* which are isomorphic to the Weyl group of $\mathcal{G}^{I\!\!C}$ [66].

In the cases of non-dominant weight Λ the character formula for the irreducible LWM is [72] :

$$ch\ L_\Lambda \ = \ \sum_{\substack{w\in W \\ w\leq w_\Lambda}} (-1)^{\ell(w_\Lambda w)}\ P_{w,w_\Lambda}(1)\ ch\ V^{w\cdot(w_\Lambda^{-1}\cdot\Lambda)}\ , \quad \Lambda\in\Gamma \tag{52}$$

where $P_{y,w}(u)$ are the Kazhdan–Lusztig polynomials $y, w \in W$ [72] (for an easier exposition see [68]), w_Λ is a unique element of W with minimal length such that the signature of $\Lambda_0 \ = \ w_\Lambda^{-1}\cdot\Lambda$ is anti-dominant or semi-anti-dominant:

$$\chi_0 \ = \ (m'_1, \ldots, m'_n), \qquad m'_k \ = \ 1 - \Lambda_0(H_k)\ \in Z\!\!\!Z_-\ . \tag{53}$$

Note that $P_{y,w}(1) \in I\!N$ for $y \leq w$.

When Λ_0 is semi-anti-dominant, i.e., at least one $m'_k = 0$, then in fact W is replaced by a reduced Weyl group W_R.

Most often the value of $P_{y,w}(1)$ is equal to 1 (as in the character formula for the finite-dimensional irreps), while the cases $P_{y,w}(1) > 1$ are related to the appearance of subsingular vectors, though the situation is more subtle, see [68].

It is interesting to see how the reducible points relevant for unitarity fit in the multiplets. In the case of d_{ij} (26) and using (13) we have:

$$m_n(d_{ij}) \ = \ 1 - 2m_j - \cdots - 2m_{n-1} - m_i - \cdots - m_{j-1}\ . \tag{54}$$

In the case of d_i (27) we have:

$$m_n(d_i) \ = \ 1 - 2m_i - \cdots - 2m_{n-1}\ . \tag{55}$$

As expected the weights related to positive energy d are not dominant $(m_n(d_{ij}) \in \mathbb{Z}_-,\ m_n(d_i) \in -\mathbb{N},\ (i < n))$, since the positive energy UIRs are infinite-dimensional. (Naturally, $m_n(d_n) = 1$ falls out of the picture since $d_n < 0$.)

Thus, the Verma modules with weights related to positive energy would be somewhere in the main multiplet (or in a reduction of the main multiplet), and the first task for calculating the character is to find the w_Λ in the character formula (52). This we do in the next subsection in the case $n = 3$.

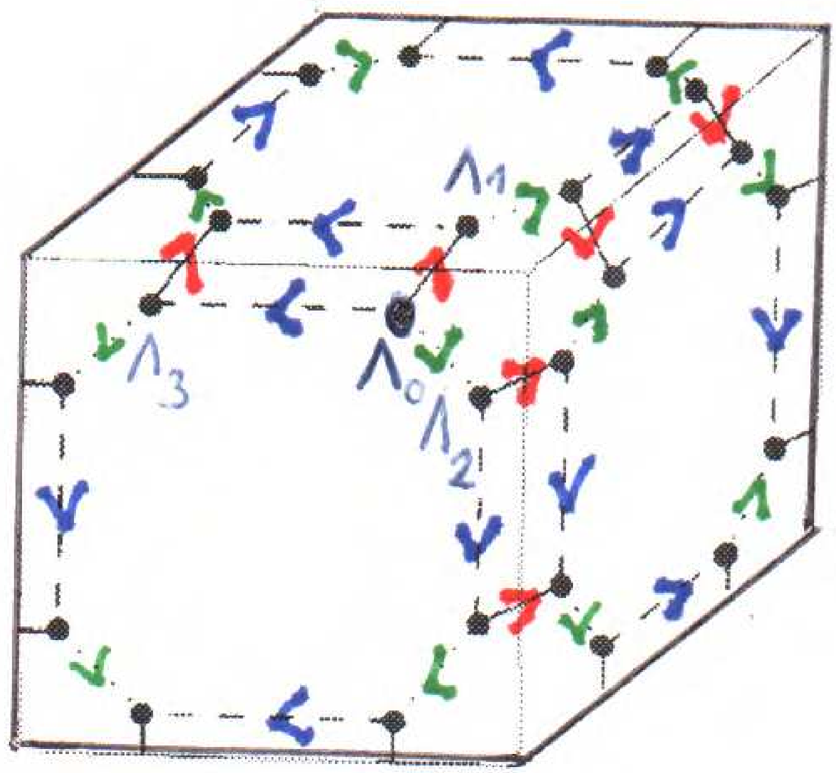
4.3 The case n=3

In order to illustrate what the main ideas we consider the first non-trivial example $n = 3$, i.e., $osp(1/6)$ actually using B_3. The Weyl group W_n of B_n has $2^n n!$ elements, i.e., 48 for B_3. Let $S = (s_1, s_2, s_3)$, $s_i \equiv s_{\alpha_i}$, be the simple reflections. They fulfill the following relations:

$$s_1^2 = s_2^2 = s_3^2 = e,\ (s_1 s_2)^3 = e,\ (s_2 s_3)^4 = e,\ s_1 s_3 \ = \ s_3 s_1\,, \tag{56}$$

e being the identity of W_3. The 48 elements may be listed as:

$$\begin{aligned}
& e\,,\ s_1\,,\ s_2\,,\ s_3 \\
& s_1s_2\,,\ s_1s_3\,,\ s_2s_1\,,\ s_2s_3\,,\ s_3s_2\,, \\
& s_1s_2s_1\,,\ s_1s_2s_3\,,\ s_1s_3s_2\,,\ s_2s_1s_3\,,\ s_2s_3s_2\,,\ s_3s_2s_1\,,\ s_3s_2s_3\,, \\
& s_1s_2s_1s_3\,,\ s_1s_2s_3s_2\,,\ s_1s_3s_2s_1\,,\ s_1s_3s_2s_3\,, \\
& s_2s_3s_2s_1\,,\ s_2s_1s_3s_2\,,\ s_3s_2s_3s_1\,,\ s_3s_2s_3s_2\,, \\
& s_1s_2s_3s_2s_1\,,\ s_1s_3s_2s_1s_3\,,\ s_1s_2s_1s_3s_2\,,\ s_1s_3s_2s_3s_2\,, \\
& s_2s_1s_3s_2s_1\,,\ s_2s_1s_3s_2s_3\,,\ s_3s_2s_3s_1s_2\,,\ s_3s_2s_3s_2s_1\,, \\
& s_1s_3s_2s_3s_2s_1\,,\ s_1s_3s_2s_1s_3s_2\,,\ s_1s_2s_1s_3s_2s_1\,,\ s_2s_1s_3s_2s_1s_3\,, \\
& s_2s_1s_3s_2s_3s_2\,,\ s_3s_2s_3s_1s_2s_1\,,\ s_3s_2s_3s_1s_2s_3\,, \\
& s_2s_1s_3s_2s_3s_2s_1\,,\ s_2s_1s_3s_2s_3s_1s_2\,,\ s_3s_2s_1s_2s_3s_2s_1\,, \\
& s_3s_2s_3s_1s_2s_1s_3\,,\ s_3s_2s_3s_1s_2s_3s_2\,, \\
& s_2s_3s_2s_1s_2s_3s_2s_1\,,\ s_3s_2s_1s_3s_2s_3s_2s_1\,,\ s_3s_2s_1s_3s_2s_3s_1s_2\,, \\
& s_2s_3s_2s_1s_3s_2s_3s_2s_1\ .
\end{aligned} \tag{57}$$



This Weyl group may be pictorially represented on a cube as in the figure, where we have given only the simple root reflections, namely, continuous (red) arrows represent action of reflection s_1, dashed (blue) arrows represent action of reflection s_2, dotted (green) arrows represent action of reflection s_3. Each face of the cube contains eight elements related by blue and green arrows representing the Weyl group of B_2 generated by s_2 and s_3. The figure contains also eight sextets (around the eight corners of the cube). Each sextet is related by red and green arrows representing the Weyl group of A_2 generated by s_1 and s_2. Finally there are 12 quartets (straddling the edges of the cube). Each quartet is formed by red and blue arrows representing the Weyl group of $A_1 \times A_1$ generated by the commuting reflections s_1 and s_3.

We use the same diagram to depict the main multiplets containing the

Verma modules V^{Λ_0} which contain (as factor module) the finite-dimensional irreps of B_3, i.e., with dominant weights Λ_0, i.e., with Dynkin labels (m_1, m_2, m_3), $m_k \in I\!N$. We may do this since these multiplets are isomorphic to the Weyl group, W_3 in our case. On the picture we have indicated the modules, Λ_0 and $\Lambda_k = s_k \cdot \Lambda_0$, $k = 1, 2, 3$. The mentioned isomorphism is fixed by assigning to Λ_0 the identity element e of W_3, and to Λ_k the reflections s_k .

The character formula for the Verma modules in our case is given explicitly by:

$$\begin{aligned} ch\ V^\Lambda &= \frac{e(\Lambda)}{(1-t_1)(1-t_2)(1-t_1t_2)} \times \\ &\times \frac{1}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3^2)} \end{aligned} \tag{58}$$

where $t_j \equiv e(\alpha_j)$.

Now we give the character formulae of the five boundary or isolated unitarity cases. Below we shall denote the signature of the dominant weight Λ_0 which determines the main multiplet by (m'_1, m'_2, m'_3), $m'_k \in I\!N$, using primes to distinguish from the signatures of the weights we are interested. We shall use also reductions of the main multiplet when the weights are semi-dominant, i.e., when some $m'_k = 0$.

• In the case of $d = d_1 = 2 + \frac{1}{2}(a_1 + a_2)$ there are twelve members of the multiplet which is a submiltiplet of a main multiplet. (Remember that that $m_1 > 1$ since $a_1 \neq 0$.) They are grouped into two standard $sl(3)$ submultiplets of six members. The first submultiplet starts from $V^{\Lambda_0^{d_1}}$, where $\Lambda_0^{d_1} = w \cdot \Lambda_0$, $w = w_{\Lambda_0^{d_1}} = s_2 s_1 s_3 s_2 s_3$, with signature:

$$\Lambda_0^{d_1} \ :\ (m_1, m_2, m'_3 = 1 - 2m_{12})\ , \quad m_1, m_2 \in I\!N\ , \quad m_{12} \equiv m_1 + m_2 \ . \tag{59}$$

The other submultiplet starts from $V^{\Lambda'_0}$ with $\Lambda'_0 = \Lambda_0^{d_1} + \delta_1 = \Lambda_0^{d_1} + \alpha_1 + \alpha_2 + \alpha_3$, with signature: $\Lambda'_0 \ :\ (m_1 - 1, m_2, m'_3 = 1 - 2m_{12})$, $m_1 > 1$. The character formula is (52) with $w_\Lambda = w_{\Lambda_0^{d_1}}$:

$$\begin{aligned} \mathrm{ch}\, \Lambda_0^{d_1} &= \frac{e(\Lambda_0^{d_1})}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3^2)} \\ &\times \{\ \mathrm{ch}\, \Lambda_{m_1,m_2}(t_1,t_2)\ -\ t_1t_2t_3\, \mathrm{ch}\, \Lambda_{m_1-1,m_2}(t_1,t_2)\ \}\ , \quad m_1 > 1 \end{aligned} \tag{60}$$

where $\mathrm{ch}\,\Lambda_{m_1,m_2}(t_1,t_2)$ is the normalized character of the finite-dimensional $sl(3)$ irrep with Dynkin labels (m_1,m_2) (and dimension $m_1m_2(m_1+m_2)/2$):

$$\mathrm{ch}\ \Lambda_{m_1,m_2}(t_1,t_2) \ = \ \frac{1-t_1^{m_1}-t_2^{m_2}+t_1^{m_1}t_2^{m_{12}}+t_1^{m_{12}}t_2^{m_2}-t_1^{m_{12}}t_2^{m_{12}}}{(1-t_1)(1-t_2)(1-t_1t_2)} \tag{61}$$

Naturally, the latter formula is a polynomial in t_1,t_2, e.g., $\mathrm{ch}\,\Lambda_{1,1}(t_1,t_2) \ = \ 1$. Note that (60) trivializes for $m_1=1$ since the second term disappears by the formal substitution: $\mathrm{ch}\ \Lambda_{0,m_2}(t_1,t_2) \ = \ 0$.

• In the case of $d=d_{12}=\frac{1}{2}(3+a_2)$ which is relevant for unitarity, i.e., $m_1=1$, there are again twelve members of the multiplet. The corresponding signature is:

$$\Lambda_0^{d_{12}} \ : \ (1,m_2,m_3'=-2m_2) \ , \quad m_2\in I\!N \ . \tag{62}$$

The multiplet is submultiplet of a reduced multiplet with 24 members obtained from a main multiplet for $m_3'=0$. As above our multiplet consists of two standard $sl(3)$ submultiplets of six members. The first submultiplet starts from $V^{\Lambda_0^{d_{12}}}$, where $\Lambda_0^{d_{12}} \ = \ w\cdot\Lambda_0$, $w \ = \ w_{\Lambda_0^{d_{12}}} \ = \ s_3s_2s_1$. The other submultiplet starts from $V^{\Lambda_0'}$ with $\Lambda_0' \ = \ \Lambda_0^{d_{12}}+m_2(\alpha_1+2\alpha_2+2\alpha_3) \ = \ \Lambda_0^{d_{12}}+m_2(\delta_1+\delta_2)$ with signature: $\Lambda_0' \ : \ (1,m_2-1,-2m_2)$. The character formula is (52), with $W \ \mapsto W_R$, (where W_R is a reduced 24-member Weyl group) and with $w_\Lambda \ = \ w_{\Lambda_0^{d_{12}}}$:

$$\mathrm{ch}\,\Lambda_0^{d_{12}} = \frac{e(\Lambda_0^{d_{12}})}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3^2)}$$
$$\times\{ \ \mathrm{ch}\,\Lambda_{1,m_2}(t_1,t_2) \ - \ (t_1t_2^2t_3^2)^{m_2}\,\mathrm{ch}\,\Lambda_{1,m_2-1}(t_1,t_2) \ \} \ , \quad m_2>1 \tag{63}$$

where $\mathrm{ch}\,\Lambda_{m_1,m_2}$ are the $sl(3)$ characters defined in (61).

• In the case of $d=d_2=1+\frac{1}{2}a_2\geq d_{13}$, i.e., $m_3'=1-2m_2$, the corresponding signature is:

$$\Lambda_0^{d_2} \ : \ (m_1,m_2,m_3'=1-2m_2) \ . \tag{64}$$

We should consider two subcases

$$1+m_1-m_2>0 \qquad \text{or} \qquad 1+m_1-m_2\leq 0$$

We start with the *first subcase* which is relevant when $d = d_2 = d_{13} = 1$ and $a_1 = a_2 = 0$, then $m_1 = m_2 = 1$, and the signature is:

$$\Lambda_0^{d_2=d_{13}} \ : \ (1, 1, -1) \ . \tag{65}$$

Our multiplet is a submultiplet of a 12-member reduced multiplet obtained when the signature of Λ_0 is $(m'_1, m'_2, m'_3) = (1, 0, 1)$, and then $\Lambda_0^{d_2=d_{13}}$ is a submodule with signature (65). Thus, we have $\Lambda_0^{d_2=d_{13}} = s_3 \cdot \Lambda_0$, i.e., $w_{\Lambda_0^{d_2=d_{13}}} = s_3$.

Explicitly, our 12-member multiplet has two $sl(3)$ submultiplets. First we take into account a $sl(3)$ sextet starting from $\Lambda_0^{d_2=d_{13}}$ with parameters $(1, 1)$. Then there is a $sl(3)$ sextet starting from $\Lambda_0^{d_2=d_{13}}+\alpha_1+2\alpha_2+3\alpha_3$ with parameters $(1, 1)$. Note that that $\alpha_1 + 2\alpha_2 + 3\alpha_3 = \delta_1 + \delta_2 + \delta_3$ is the weight of the subsingular vector (41).

The character formula is (52), with $W \mapsto W_R$, (where W_R is a reduced 12-member Weyl group) and $w_\Lambda = s_3$:

$$\begin{aligned} &\mathrm{ch}\,\Lambda_0^{d_2=d_{13}} \ = \\ &= \ \frac{e(\Lambda_0^{d_2=d_{13}})}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3^2)} \times \\ &\times \{ \ 1 \ - \ t_1t_2^2t_3^3 \ \} \end{aligned} \tag{66}$$

• In the case of $d = d_2 = 1+\frac{1}{2}a_2 > d_{13} = 1$, i.e., $m_1 = 1$, $m_2 = 1+a_2 > 1$, thus, this is the subcase $1 + m_1 - m_2 = m_{13} \leq 0$. The multiplet has 24 members for $m_2 > 2$ $(m_{13} < 0)$ and starts with $\Lambda_0^{\prime d_2} \ = \ = \ s_3\, s_2\, s_1 \cdot \Lambda_0$, with signatures:

$$\begin{aligned} &\Lambda_0 \ : \ (m_2 - 2, 1, 1) \ , \\ &\Lambda_0^{\prime d_2} \ : \ (1, m_2, m'_3 = 1 - 2m_2) \ , \quad m_2 \in 1 + I\!N \ . \end{aligned} \tag{67}$$

It has four $sl(3)$ submultiplets. First we take into account a $sl(3)$ sextet starting from $\Lambda_0^{\prime d_2}$ with parameters $(1, m_2)$. Then there is a $sl(3)$ sextet starting from $\Lambda_0^{\prime d_2} + \alpha_{23}$ with parameters $(2, m_2 - 1)$. Then there is a $sl(3)$ sextet with parameters $(2, m_2 - 2)$ starting from a Verma module $V^{\Lambda''}$, $\Lambda'' = \Lambda_0^{\prime d_2} + \alpha_1 + 3\alpha_{23}$. Finally, there is a $sl(3)$ sextet with parameters $(1, m_2{-}2)$, starting from a Verma module $V^{\Lambda'''}$, $\Lambda''' = \Lambda_0^{\prime d_2}{+}2(\alpha_1{+}2\alpha_2{+}2\alpha_3)$.

We have the *Conjecture* that the character formula is (52) and $w_\Lambda = s_3\, s_2\, s_1$:

$$\mathrm{ch}\,\Lambda_0^{\prime d_2} = \frac{e(\Lambda_0^{\prime d_2})}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3^2)}$$

$$\times \{\ \mathrm{ch}\,\Lambda_{1,m_2}(t_1,t_2)\ -\ t_2t_3\,\mathrm{ch}\,\Lambda_{2,m_2-1}(t_1,t_2)\ +$$

$$+\ t_1t_2^3t_3^3\,\mathrm{ch}\,\Lambda_{2,m_2-2}(t_1,t_2)\ -\ t_1^2t_2^4t_3^4\,\mathrm{ch}\,\Lambda_{1,m_2-2}(t_1,t_2)\ \} \qquad (68)$$

When $m_2 = 2$ ($a_2 = 1$, $m_{13} = 0$) the weight Λ_0 is semi dominant, the main multiplet reduces to 24 members, our multiplet reduces to only 12 members, consisting of the first two $sl(3)$ submultiplets mentioned above. The character formula takes this into account by construction since for $m_2 = 2$ the terms in the 2nd row are automatically zero (due to the fact that the $sl(3)$ character formula gives zero: $\mathrm{ch}\,\Lambda_{1,0}(t_1,t_2) = 0$).

• In the case of $d = d_{23} = \frac{1}{2}$, $a_1 = a_2 = 0$, i.e., $m_1 = m_2 = 1$, and the signature is:

$$\Lambda_0^{d_{23}}\ :\ (1,1,0)\ . \qquad (69)$$

This is in fact a multiplet with 24 members which is reduction of the main multiplet starting with the semi dominant weight (69).

The multiplet consists of four $sl(3)$ submultiplets. First there is a $sl(3)$ sextet starting from $\Lambda_0^{d_{23}}$ with parameters $(1,1)$. Then a $sl(3)$ sextet starting from $\Lambda_0^{d_{23}} + \alpha_2 + 2\alpha_3$ with parameters $(2,1)$. Then a $sl(3)$ sextet starting from $\Lambda_0^{d_{23}} + \alpha_1 + 2\alpha_2 + 4\alpha_3$ with parameters $(1,2)$. Then a $sl(3)$ sextet starting from $\Lambda_0^{d_{23}} + 2\alpha_1 + 4\alpha_2 + 6\alpha_3$ with parameters $(1,1)$.

The character formula is (52), however, with $W \mapsto W_R$, where W_R is the reduced 24-member Weyl group, (generated by $s_1, s_2, s_3s_2s_3$) and $w_\Lambda =$

1 :

$$\mathrm{ch}\,\Lambda_0^{d_{23}} \;=\; \frac{e(\Lambda_0^{d_{23}})}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3^2)}$$

$$\times\ \{\ 1\ -\ t_2t_3^2\,\mathrm{ch}\,\Lambda_{2,1}(t_1,t_2)\ +$$

$$+\ t_1t_2^2t_3^4\,\mathrm{ch}\,\Lambda_{1,2}(t_1,t_2)\ -\ t_1^2t_2^4t_3^6\ \}\ =$$

$$=\ \frac{e(\Lambda_0^{d_{23}})}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3^2)}$$

$$\times\ \{\ 1\ -\ t_2t_3^2\,(1+t_1+t_1t_2)\ +$$

$$+\ t_1t_2^2t_3^4\,(1+t_2+t_1t_2)\ -\ t_1^2t_2^4t_3^6\ \}\ =$$

$$=\ \frac{e(\Lambda_0^{d_{23}})}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)} \tag{70}$$

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